Intrinsically triple-linked graphs in $\mathbb{R}P^3$ *

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Abstract

Flapan, *et al* [8] showed that every spatial embedding of K_{10}, the complete graph on ten vertices, contains a non-split three-component link (K_{10} is *intrinsically triple-linked*). The papers [2] and [7] extended the list of known intrinsically triple-linked graphs in $\mathbb{R}^3$ to include several other families of graphs. In this paper, we will show that while some of these graphs can be embedded 3-linklessly in $\mathbb{R}P^3$, K_{10} is intrinsically triple-linked in $\mathbb{R}P^3$.

1 Introduction

Real projective 3-space, $\mathbb{R}P^3$, is defined to be the quotient $S^3/\sim$, where $\sim$ is the antipodal relation $x \sim -x$ and can be thought of as the disk, D^3, with antipodal boundary points identified. Projective space has a non-trivial first homology group, $H_1 \cong \mathbb{Z}/2\mathbb{Z}$. The generator for the group, g, is the cycle originating from the line in D^3 that runs between the north and south poles. Mroczkowski [12] has shown that every knot in $\mathbb{R}P^3$ can be transformed into either the trivial cycle or g by crossing changes and generalized Reidemeister moves on an $\mathbb{R}P^2$ projection of the knot. Thus, there are two non-equivalent unknots in $\mathbb{R}P^3$. Cycles that can be "unknotted" into a cycle homologous to g will be referred to as *1-homologous cycles*. Cycles that can be "unknotted" into a trivial cycle will be referred to as *0-homologous cycles*.

A *link* in $\mathbb{R}P^3$ is *splittable* if one of the components can be contained within a sphere, embedded in the space, while the other component remains in the complement

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of the sphere. Otherwise, the link in $\mathbb{R}P^3$ is *non-split*. A non-split link can be formed one of three ways in $\mathbb{R}P^3$: two 0-homologous cycles, a 0-homologous cycle with a 1-homologous cycle, and two 1-homologous cycles. Note: Two disjoint 1-homologous cycles will always form a non-split link. Similarly, a *non-split triple-link* is a non-split link of three components. In this paper we will refer to non-split linked cycles as *linked cycles* and an embedding of a graph as *linked* if it contains a non-split link. We will refer to a non-split triple-link as a *triple-link* and an embedding of a graph as *triple-linked* if it contains a non-split triple-link.

A graph H is a *minor* of a graph G if H can be obtained from G through a series of vertex removals, edge removals, or edge contractions. A graph G is *minor-minimal* with respect to a property P if G has property P but no minor of G has property P.

If G is a graph, define an *induced subgraph*, $G[v_1, v_2, ..., v_n]$, of G to be the subgraph of G on vertices $\{v_1, v_2, ..., v_n\}$ and the set of edges in G with both endpoints in the set $\{v_1, v_2, ..., v_n\}$.

A graph G is *intrinsically linked in* $\mathbb{R}^3$ if and only if G contains a non-split link in every spatial embedding. We define *intrinsically linked in* $\mathbb{R}P^3$ analogously. It has been shown that the complete set of minor-minimal intrinsically linked graphs in $\mathbb{R}^3$ is the set of Petersen Family graphs [14] (including K_6 and graphs obtained from K_6 by $\Delta - Y$ and $Y - \Delta$ exchanges). However, all Petersen Family graphs except for $K_{4,4} - e$ embed linklessly in $\mathbb{R}P^3$ [3]. While [3] characterizes several families of graphs that are minor-minimally intrinsically linked in $\mathbb{R}P^3$, the complete set of minor-minally intrinsically linked graphs in $\mathbb{R}P^3$, which is finite due to the result in [13], remains to be found.

A graph G is *intrinsically triple-linked in* $\mathbb{R}^3$ if and only if G contains a non-split link of three components in every spatial embedding. We define *intrinsically triple-linked in* $\mathbb{R}P^3$ analogously. An embedding is said to be 3-*linkless* if and only if it does not contain a triple-link.

While Conway, Gordon [4], and Sachs [15, 16] showed that K_6 is intrinsically linked in $\mathbb{R}^3$, K_6 can be linklessly embedded in $\mathbb{R}P^3$; it has been shown that 7 is the smallest n for which K_n is intrinsically linked in $\mathbb{R}P^3$ [3]. In contrast, while 10 was shown to be the smallest n for which K_n is intrinsically triple-linked in $\mathbb{R}^3$ [9], we have shown that 10 is also the smallest n for which K_n is intrinsically triple-linked in $\mathbb{R}P^3$. It remains to be shown whether K_{10} is minor-minimal with respect to triple-linking in $\mathbb{R}P^3$. Additionally, we have shown two other intrinsically triple-linked graphs in $\mathbb{R}^3$ can be embedded without a triple-link in $\mathbb{R}P^3$. A complete set of minor-minimal intrinsically triple-linked graphs remains to be found, in both $\mathbb{R}^3$ and $\mathbb{R}P^3$. Such sets are finite due to the result in [13].

2 Intrinsically triple-linked complete graphs on n vertices

We will need the following lemmas:

Lemma 1. [3] *The graphs obtained by removing two edges from K_7 and removing one edge from $K_{4,4}$ are intrinsically linked in $\mathbb{R}P^3$.*

Lemma 2. [3] *Given a linkless embedding of K_6 in $\mathbb{R}P^3$, no K_4 subgraph can have all 0-homologous cycles.*

We also use the following elementary observation.

Lemma 3. *For every embedding into $\mathbb{R}P^3$, K_4 has an even number of 1-homologous cycles.*

The following lemma was shown true in $\mathbb{R}^3$ by Flapan, Naimi, and Pommersheim [9] and the proof holds true analogously in $\mathbb{R}P^3$.

Lemma 4. *Let G be a graph embedded in $\mathbb{R}P^3$ that contains cycles C_1, C_2, C_3 and C_4. Suppose C_1 and C_4 are disjoint from each other and from C_2 and C_3 and suppose $C_2 \cap C_3$ is a simple path. If $lk(C_1, C_2) \neq 0$ and $lk(C_3, C_4) \neq 0$, then G contains a non-split three-component link.*

The following proposition is not the main result of this paper. However, the proof is included because it is concise and since its method does not hold for proving K_{10} is also triple-linked.

Proposition 5. *The graph K_{11} is intrinsically triple-linked in $\mathbb{R}P^3$.*

Proof. Let G be a complete graph on the vertex set $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11\}$. Embed G in $\mathbb{R}P^3$. Consider $G[1, 2, 3, 4, 5, 6, 7]$. Since K_7 is intrinsically linked in $\mathbb{R}P^3$, this subgraph contains a pair of linked cycles that can be reduced to two linked 3-cycles. Without loss of generality, let $C_1 = (1, 2, 3)$ and $C_2 = (4, 5, 6)$ be the pair of linked cycles in $G[1, 2, 3, 4, 5, 6, 7]$.

Now consider $G[5, 6, 7, 8, 9, 10, 11]$. Since K_7 is intrinsically linked in $\mathbb{R}P^3$, this subgraph contains a pair of linked cycles that can be reduced to two linked 3-cycles. In $G[5, 6, 7, 8, 9, 10, 11]$, one cycle must use $\{v_5\}$ and the other cycle must use $\{v_6\}$, or Lemma 4 would apply immediately. Without loss of generality, let $C_3 = (5, 7, 9)$ and $C_4 = (6, 8, 10)$ be the pair of linked cycles in $G[5, 6, 7, 8, 9, 10, 11]$.

Consider $G[1, 2, 3, 4, 6, 11]$. By Lemma 2, $G[1, 2, 3, 11]$ must contain a 1-homologous cycle or $G[1, 2, 3, 4, 6, 11]$ contains a pair of linked cycles and Lemma 4 applies with C_3

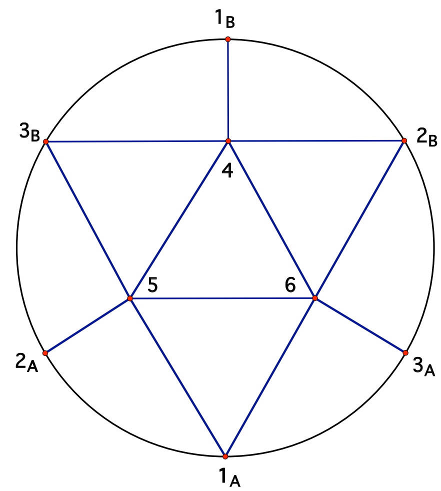

Figure 1: A projection of a linkless embedding of K_6 in $\mathbb{R}P^3$.

and C_4. Thus by Lemma 3, two cycles in $A = \{(1,2,3),(1,2,11),(1,3,11),(2,3,11)\}$ must be 1-homologous 3-cycles.

Now consider $G[6,7,8,9,10,11]$. By Lemma 2, $G[7,8,9,10]$ must contain a 1-homologous cycle or $G[6,7,8,9,10,11]$ contains a pair of linked cycles and Lemma 4 applies with C_1 and C_2. Thus by Lemma 3, two cycles in $B = \{(7,8,9),(7,8,10),(7,9,10),(8,9,10)\}$ must be 1-homologous 3-cycles.

Since every cycle in A is disjoint from every cycle in B, and at least two cycles in each set are 1-homologous, there exists a link using one cycle from A and one cycle from B. Lemma 4 then applies since every cycle in A shares at least a simple path with C_1, and C_2 and the cycle from B are disjoint from each other, C_1, and the cycle from A. Thus, G contains a triple-link.

$\square$

Proposition 6. *If G is K_6 embedded in $\mathbb{R}P^3$ and G has two disjoint 0-homologous cycles, then G contains a non-split link.*

Proof. Assume G can be embedded so that it has two disjoint 0-homologous cycles and so that it does not have a non-split link. Without loss of generality, let $(1,2,3)$ and $(4,5,6)$ be 0-homologous cycles in G. Consider $G[1,2,3,4]$. Since G is not linked, by Lemma 2 and Lemma 3, $G[1,2,3,4]$ must have two 1-homologous cycles. Without loss of generality, let $(1,2,4)$ and $(1,3,4)$ be 1-homologous cycles. Similarly, $G[2,4,5,6]$

must also have two 1-homologous cycles. Since $(4,5,6)$ is 0-homologous by assumption and $(2,5,6)$ is disjoint from $(1,3,4)$, $(2,4,5)$ and $(2,4,6)$ are 1-homologous cycles. Similarly, $G[1,2,3,6]$ has two 1-homologous cycles. Since $(1,2,3)$ is 0-homologous by assumption and $(1,3,6)$ is disjoint from $(2,4,5)$, $(1,2,6)$ and $(2,3,6)$ are 1-homologous cycles or G would contain a pair of linked cycles. Now consider $G[1,3,5,6]$, which must also have two 1-homologous cycles by Lemma 2 and Lemma 3. Since $(1,3,5)$ is disjoint from $(2,4,6)$, $(1,3,6)$ is disjoint from $(2,4,5)$, and $(3,5,6)$ is disjoint from $(1,2,4)$, $(1,3,5)$, $(1,3,6)$, and $(3,5,6)$ must be 0-homologous. This forces $G[1,3,5,6]$ to contain only 0-homologous cycles, and thus G is linked by 2. Thus, G cannot have two disjoint 0-homologous cycles and not be linked.

□

Proposition 7. *Up to ambient isotopy and crossing changes, Figure 1 is the only way to linklessly embed K_6 in $\mathbb{R}P^3$.*

Proof. Let G be a complete graph on the vertex set $\{1, 2, 3, 4, 5, 6\}$. Embed G in $\mathbb{R}P^3$.

The graph G has a 0-homologous 3-cycle, else G has disjoint 1-homologous cycles and is thus linked by Proposition 6. Without loss of generality, let $(4,5,6)$ be a 0-homologous 3-cycle. Now consider vertices $\{1, 2, 3\}$. If $(1,2,3)$ is 0-homologous, G is linked; thus, we assume $(1,2,3)$ is a 1-homologous cycle. Mroczkowski [12] showed that any cycle can be made into an unknotted 0- or 1-homologous cycle by crossing changes, so we can assume after crossing changes and ambient isotopy the embedding has a projection as drawn in Figure 1 (except the edges between the vertices $\{1,2,3\}$ and $\{4,5,6\}$ may be more complicated than in the Figure) with vertices $\{1,2,3\}$ on the boundary and the edges between them on the boundary.

We may use ambient isotopy and crossing changes so that edges from $\{1,2,3\}$ to $\{4,5,6\}$ connect in the projection without crossing the boundary of D^2. We now show that we may connect them, without loss of generality, as depicted in Figure 1.

If vertex $v \in \{1,2,3\}$, v must connect to at least one of $\{4, 5, 6\}$ from v_A and to at least one of $\{4, 5, 6\}$ from v_B, else there would be a 0-homologous K_4 and G would be linked by Lemma 2. Without loss of generality, assume v_2 connects to v_4 and v_6 from v_{2_B} and to v_5 from v_{2_A}.

If v_1 connects to v_4 and v_6 from v_{1_B}, then $G[1,2,4,6]$ is a 0-homologous K_4 and G is linked by Lemma 2. Thus, v_1 connects to either v_4 or v_6 from v_{1_B} and connects to the other from v_{1_A}. Without loss of generality, let v_{1_B} connect to v_4; so, v_{1_A} connects to v_6. If v_{1_B} connects to v_5, then either v_{3_A} or v_{3_B} must connect to both v_5 and v_6

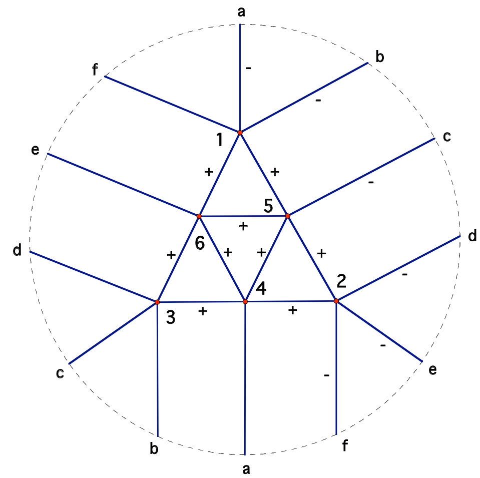

Figure 2: A signed linkless embedding of K_6 in $\mathbb{R}P^3$.

and the other to v_4, else G has a 0-homologous K_4. Without loss of generality, let v_{3_A} connect to v_5 and v_6 and v_{3_B} connect to v_4. Then, $(1,2,5)$ and $(3,4,6)$ are disjoint 1-homologous cycles so G is linked. Thus, v_{1_A} connects to v_5. Now, if v_{3_A} connects to either v_4 and v_6 or v_5 and v_6, then G has a 0-homologous K_4 and is linked by Lemma 2. So, v_{3_A} must connect to v_6 and v_{3_B} must connect to v_4 and v_5.

□

Signed graphs, that is, graphs with each edge assigned a $+$ or a $-$ sign, have been studied extensively and were first introduced by Harary [10], see also [17]. An embedding of a graph G into $\mathbb{R}P^3$ induces a signed graph of G as follows: deform the embedding so that no vertices touch the line at infinity and all intersections of edges with the line at infinity are transverse. Assign $+$ edges to be edges that hit the boundary an even number of times and $-$ edges to be edges that hit the boundary an odd number of times. If a cycle has an odd number of $-$ edges, then the cycle is 1-homologous. Two embeddings, G_1 and G_2, of a graph G are *crossing-change equivalent* if and only if G_1 can be obtained from G_2 by crossing changes and ambient isotopy. Thus, by Proposition 7, a linkless K_6 is crossing-change equivalent to the embedding in Figure 2. That is, $(1,2)$, $(1,3)$, $(2,3)$, $(1,4)$, $(2,5)$, and $(3,6)$ are $-$ edges, and the other nine edges are $+$ edges.

Theorem 8. *The graph K_{10} is intrinsically triple-linked in $\mathbb{R}P^3$.*

Proof. Let G be a graph isomorphic to K_{10} on the vertex set $\{1,2,3,4,5,6,7,8,9,10\}$. Embed G in $\mathbb{R}P^3$. Assume the embedding is 3-linkless.

If every subgraph of G isomorphic to K_6 is linked, then Flapan, Naimi, and Pommersheim's proof [9] that K_{10} is intrinsically linked in $\mathbb{R}^3$ nearly works, except

they do use the fact that $K_{3,3,1}$ is intrinsically linked at the very end and $K_{3,3,1}$ is not intrinsically linked in $\mathbb{R}P^3$. Bowlin-Foisy [2], however, modify [9] slightly so that only the fact that K_6 is intrinsically linked is needed. Thus, in the case that every subgraph of G isomorphic to K_6 is linked, then G is triple-linked. So we may assume there exists a linkless K_6 subgraph in G. Without loss of generality, assume this linkless K_6 is on vertices $\{1, 2, 3, 4, 5, 6\}$. By Proposition 7, this K_6 has an embedding crossing-change equivalent to that in drawn in Figure 2.

Claim: The embedded induced subgraph $G[7, 8, 9, 10]$ is 0-homologous.

Proof. Assume $G[7, 8, 9, 10]$ has a 1-homologous cycle. Without loss of generality, let $(7, 8, 9)$ be a 1-homologous cycle. Now consider $G[4, 5, 6, 10]$. If $G[4, 5, 6, 10]$ is not 0-homologous, then two of $(4, 5, 10)$, $(4, 6, 10)$, and $(5, 6, 10)$ are 1-homologous by Lemma 3. Then $(1, 2, 3)$, $(7, 8, 9)$, and a cycle from $G[4, 5, 6, 10]$ comprise three disjoint 1-homologous cycles, so G is triple-linked. Thus, $G[4, 5, 6, 10]$ is 0-homologous and so $G[1, 2, 4, 5, 6, 10]$ has a pair of linked cycles by Lemma 2. Since $(7, 8, 9)$ is 1-homologous, and $(7, 8, 9)$ is disjoint from all the 1-homologous cycles in the second column of Table 1, Lemma 4 applies and G has a triple-link. Thus, $G[7, 8, 9, 10]$ is 0-homologous.

Possible Linked Cycles in $G[1, 2, 4, 5, 6, 10]$	1-Homologous Cycle that shares an edge with a linked cycle
$(1, 2, 4)$, $(5, 6, 10)$	$(1, 2, 3)$
$(1, 2, 5)$, $(4, 6, 10)$	$(1, 2, 3)$
$(1, 2, 6)$, $(4, 5, 10)$	$(1, 2, 3)$
$(1, 2, 10)$, $(4, 5, 6)$	$(1, 2, 3)$
$(1, 4, 5)$, $(2, 6, 10)$	$(1, 3, 5)$
$(1, 4, 6)$, $(2, 5, 10)$	$(1, 4, 6)$
$(1, 4, 10)$, $(2, 5, 6)$	$(2, 5, 6)$
$(1, 5, 6)$, $(2, 4, 10)$	$(1, 3, 5)$
$(1, 5, 10)$, $(2, 4, 6)$	$(1, 3, 5)$
$(1, 6, 10)$, $(2, 4, 5)$	$(2, 4, 5)$

Table 1.

□

Since $G[7, 8, 9, 10]$ is 0-homologous, we may assume all edges in $G[7, 8, 9, 10]$ are $+$ edges. The edges in $G[1, 2, 3, 4, 5, 6]$ are $+$ and $-$ edges as defined in Figure 2. The following arguments will use this modified embedding of G, however, since ambient

isotopy and crossing changes do not change the homology of the cycles, the linking arguments will still hold for the original embedding. Similar to the argument highlighted in Table 1, many of the following arguments rely on K_6 subgraphs of G that must have a pair of linked cycles. The modified embedding may have a different pair of linked cycles in the subgraph than in the original embedding, however a pair of linked cycles still exists and the argument does not rely on which cycles are linked. We now consider the signs of the edges connecting $G[1,2,3,4,5,6]$ to $G[7,8,9,10]$.

Claim: If $v \in \{1,2,3\}$, then edges from v to $G[7,8,9,10]$ must all be $+$ edges or all $-$ edges.

Proof. Assume vertex v_1 does not connect by all $+$ edges or all $-$ edges to $G[7,8,9,10]$. Without loss of generality, let $(1,7)$ be a $+$ edge and $(1,8)$ be a $-$ edge. Then, $(1,7,8)$ is a 1-homologous cycle. Consider $G[3,4,6,9]$. Since $(3,4,6)$ is a 1-homologous cycle, $G[3,4,6,9]$ must have another 1-homologous cycle by Lemma 3. If $(3,4,9)$ is 1-homologous, then $(1,7,8)$, $(2,5,6)$, and $(3,4,9)$ form three disjoint 1-homologous cycles, so G is triple-linked. If $(3,6,9)$ is 1-homologous, then $(1,7,8)$, $(2,4,5)$, and $(3,6,9)$ form three disjoint 1-homologous cycles, so G is triple-linked. Thus, $(4,6,9)$ is a 1-homologous cycle.

Now consider $G[2,3,4,9]$. Since $(2,3,4)$ is a 1-homologous cycle, $G[2,3,4,9]$ must have another 1-homologous cycle by Lemma 3. If $(3,4,9)$ is 1-homologous, then $(1,7,8)$, $(2,5,6)$, and $(3,4,9)$ form three disjoint 1-homologous cycles, so G is triple-linked. If $(2,4,9)$ is 1-homologous, then $(1,7,8)$, $(2,4,9)$, and $(3,5,6)$ form three disjoint 1-homologous cycles, so G is triple-linked. Thus, $(2,3,9)$ is a 1-homologous cycle.

Similarly, consider $G[3,5,6,9]$. Since $(3,5,6)$ is a 1-homologous cycle, $G[3,5,6,9]$ must have another 1-homologous cycle by Lemma 3. If $(3,6,9)$ is 1-homologous, then $(1,7,8)$, $(2,4,5)$, and $(3,6,9)$ form three disjoint 1-homologous cycles, so G is triple-linked. If $(5,6,9)$ is 1-homologous, then $(1,7,8)$, $(2,3,4)$, and $(5,6,9)$ form three disjoint 1-homologous cycles, so G is triple-linked. Thus, $(3,5,9)$ is a 1-homologous cycle.

Since $(1,7,8)$ and $(4,6,9)$ are 1-homologous, $G[2,3,5,10]$ is 0-homologous or else there are three disjoint 1-homologous cycles. Thus, $G[2,3,4,5,6,10]$ has a pair of linked cycles by Lemma 2. Since $(1,7,8)$ is 1-homologous, and $(1,7,8)$ is disjoint from all the 1-homologous cycles in the second column of Table 2, Lemma 4 applies and G has a pair of links. Thus, vertex v_1 must have all $+$ edges or all $-$ edges to $G[7,8,9,10]$. Similar reasoning applies to vertices v_2 and v_3.

Possible Linked Cycles in $G[2,3,4,5,6,10]$	1-Homologous Cycle that shares an edge with a linked cycle
$(2,3,4)$, $(5,6,10)$	$(2,3,4)$
$(2,3,5)$, $(4,6,10)$	$(4,6,9)$
$(2,3,6)$, $(4,5,10)$	$(2,3,9)$
$(2,3,10)$, $(4,5,6)$	$(2,3,9)$
$(2,4,5)$, $(3,6,10)$	$(2,4,5)$
$(2,4,6)$, $(3,5,10)$	$(4,6,9)$
$(2,4,10)$, $(3,5,6)$	$(3,5,9)$
$(2,5,6)$, $(3,4,10)$	$(2,5,6)$
$(2,5,10)$, $(3,4,6)$	$(4,6,9)$
$(2,6,10)$, $(3,4,5)$	$(3,5,9)$

Table 2.

□

Claim: If $v \in \{4,5,6\}$, then edges from v to $G[7,8,9,10]$ must all be $+$ edges or all $-$ edges.

Proof. Assume vertex v_4 does not have all $+$ edges or all $-$ edges to $G[7,8,9,10]$. Without loss of generality, let $(4,7)$ be a $+$ edge and $(4,8)$ be a $-$ edge. Then, $(4,7,8)$ is a 1-homologous cycle. Consider $G[1,2,3,9]$. Since $(1,2,3)$ is a 1-homologous cycle, $G[1,2,3,9]$ must have another 1-homologous cycle by Lemma 3. If $(1,3,9)$ is 1-homologous, then $(1,3,9)$, $(2,5,6)$, and $(4,7,8)$ form three disjoint 1-homologous cycles, so G is triple-linked. If $(1,2,9)$ is 1-homologous, then $(1,2,9)$, $(3,5,6)$, and $(4,7,8)$ form three disjoint 1-homologous cycles, so G is triple-linked. Thus, $(2,3,9)$ is a 1-homologous cycle.

Now consider $G[2,5,6,9]$. Since $(2,5,6)$ is a 1-homologous cycle, $G[2,5,6,9]$ must have another 1-homologous cycle by Lemma 3. If $(2,6,9)$ is 1-homologous, then $(1,3,5)$, $(2,6,9)$, and $(4,7,8)$ form three disjoint 1-homologous cycles, so G is triple-linked. If $(5,6,9)$ is 1-homologous, then $(1,2,3)$, $(4,7,8)$, and $(5,6,9)$ form three disjoint 1-homologous cycles, so G is triple-linked. Thus, $(5,6,9)$ is a 1-homologous cycle.

Since $(2,3,9)$ and $(4,7,8)$ are 1-homologous, $G[1,5,6,10]$ is 0-homologous or else there are three disjoint 1-homologous cycles. Thus, by Lemma 2, $G[1,2,3,5,6,10]$ has a pair of linked cycles. Since $(4,7,8)$ is 1-homologous, and $(4,7,8)$ is disjoint from all the 1-homologous cycles in the second column of Table 3, Lemma 4 applies

and G has a pair of links. Thus, vertex v_4 must have all + edges or all − edges to $G[7,8,9,10]$. Similarly, vertices v_5 and v_6 must have all + edges or all − edges to $G[7,8,9,10]$.

Possible Linked Cycles in $G[1,2,3,5,6,10]$	1-Homologous Cycle that shares an edge with a linked cycle
$(1,2,3)$, $(5,6,10)$	$(1,2,3)$
$(1,2,5)$, $(3,6,10)$	$(2,5,9)$
$(1,2,6)$, $(3,5,10)$	$(1,2,6)$
$(1,2,10)$, $(3,5,6)$	$(3,5,6)$
$(1,3,5)$, $(2,6,10)$	$(1,3,5)$
$(1,3,6)$, $(2,5,10)$	$(2,5,9)$
$(1,3,10)$, $(2,5,6)$	$(2,5,9)$
$(1,5,6)$, $(2,3,10)$	$(2,3,9)$
$(1,5,10)$, $(2,3,6)$	$(2,3,9)$
$(1,6,10)$, $(2,3,5)$	$(2,3,9)$

Table 3.

□

Since each vertex in $G[1,2,3,4,5,6]$ has either all + edges or all − edges to $G[7,8,9,10]$, there are 2^6 possible embedding classes, given our restrictions on how $G[1,2,3,4,5,6]$ and $G[7,8,9,10]$ are embedded. We consider all the cases. Note: If vertex v_1 connects to $G[7,8,9,10]$ with all + edges, we write v_{1+}, else we write v_{1-}.

Consider the embedding of G with v_{1+} and v_{4+}.

If we have one of the following embeddings: v_{2+}, v_{3+}, v_{5+}, and v_{6+}; v_{2+}, v_{3-}, v_{5+}, and v_{6-}; v_{2-}, v_{3+}, v_{5-}, and v_{6+}; v_{2-}, v_{3-}, v_{5-}, and v_{6-}, then $(1,4,7)$, $(2,5,8)$, and $(3,6,9)$ form three disjoint 1-homologous cycles, so G has a triple-link.

If we have one of the following embeddings: v_{2+}, v_{3+}, v_{5+}, and v_{6-}; v_{2+}, v_{3+}, v_{5-}, and v_{6+}; v_{2-}, v_{3-}, v_{5-}, and v_{6+}; v_{2-}, v_{3-}, v_{5+}, and v_{6-}, then $(1,4,7)$, $(2,3,8)$, and $(5,6,9)$ form three disjoint 1-homologous cycles, so G has a triple-link.

If we have one of the following embeddings: v_{2+}, v_{3+}, v_{5-}, and v_{6-}; v_{2-}, v_{3-}, v_{5+}, and v_{6+}, then $(1,4,7)$, $(2,6,8)$, and $(3,5,9)$ form three disjoint 1-homologous cycles, so G has a triple-link.

If the embedding is v_{2-}, v_{3+}, v_{5+}, v_{6-}, since $(1,4,7)$ and $(5,6,8)$ are 1-homologous cycles, $G[2,3,9,10]$ must be 0-homologous or G is triple-linked, so, by Lemma 2, $G[1,2,3,4,9,10]$ has a pair of links in this embedding class. Since $(5,6,8)$ is 1-homologous, and $(5,6,8)$ is disjoint from all the 1-homologous cycles in the second

column of Table 4, Lemma 4 applies and G has a pair of links. If the embedding is v_{2+}, v_{3-}, v_{5-}, v_{6+}, G is linked by a similar argument.

Possible Linked Cycles in $G[1,2,3,4,9,10]$	1-Homologous Cycle that shares an edge with a linked cycle
$(1,2,3)$, $(4,9,10)$	$(1,2,3)$
$(1,2,4)$, $(3,9,10)$	$(1,4,7)$
$(1,2,9)$, $(3,4,10)$	$(1,2,7)$
$(1,2,10)$, $(3,4,9)$	$(1,2,7)$
$(1,3,4)$, $(2,9,10)$	$(1,4,7)$
$(1,3,9)$, $(2,4,10)$	$(1,3,7)$
$(1,3,10)$, $(2,4,9)$	$(1,3,7)$
$(1,4,9)$, $(2,3,10)$	$(1,4,7)$
$(1,4,10)$, $(2,3,9)$	$(1,4,7)$
$(1,9,10)$, $(2,3,4)$	$(2,3,4)$

Table 4.

If the embedding is v_{2+}, v_{3-}, v_{5+}, and v_{6+}, since $G[7,8,9,10]$ is 0-homologous, by Lemma 2, $G[1,4,7,8,9,10]$ has a pair of links. Since $(3,5,6)$ is 1-homologous, and $(3,5,6)$ is disjoint from all the 1-homologous cycles in the second column of Table 5, Lemma 4 applies and G has a pair of links. If we have one of the following embeddings: v_{2-}, v_{3+}, v_{5-}, and v_{6-}; v_{2+}, v_{3-}, v_{5-}, and v_{6-}; v_{2-}, v_{3+}, v_{5+}, and v_{6+}, then G is linked by a similar argument.

Possible Linked Cycles in $G[1,4,7,8,9,10]$	1-Homologous Cycle that shares an edge with a linked cycle
$(1,4,7)$, $(8,9,10)$	$(1,4,7)$
$(1,4,8)$, $(7,9,10)$	$(1,4,8)$
$(1,4,9)$, $(7,8,10)$	$(1,4,9)$
$(1,4,10)$, $(7,8,9)$	$(1,4,10)$
$(1,7,8)$, $(4,9,10)$	$(1,2,7)$
$(1,7,9)$, $(4,8,10)$	$(1,2,7)$
$(1,7,10)$, $(4,8,9)$	$(1,2,7)$
$(1,8,9)$, $(4,7,10)$	$(1,2,8)$
$(1,8,10)$, $(4,7,9)$	$(1,2,8)$
$(1,9,10)$, $(4,7,8)$	$(1,2,9)$

Table 5.

This list exhausts the possible embeddings if both we have both v_{1+} and v_{4+}. The same arguement holds if the embedding is v_{1-} and v_{4-}. Thus, we can now assume the edges from v_1 and v_4 to $G[7,8,9,10]$ have different signs.

Consider the embedding of G with v_{1+} and v_{4-}. We can assume that the pairs $\{3,6\}$, and $\{2,5\}$ have different signs or the same arguments for v_1 and v_4 with the same sign holds from above.

If the embedding is v_{2+}, v_{3+}, v_{5-}, and v_{6-}, then $(1,6,7)$, $(3,5,8)$, and $(2,4,9)$ form three 1-homologous cycles, so G has a triple-link.

If the embedding is v_{2+}, v_{3-}, v_{5-}, and v_{6+}, since $G[7,8,9,10]$ is 0-homologous, by Lemma 2, $G[4,6,7,8,9,10]$ has a pair of links. Since $(1,2,3)$ is 1-homologous, and $(1,2,3)$ is disjoint from all the 1-homologous cycles in the second column of Table 6, Lemma 4 applies and G has a pair of links. A similar argument holds if the embedding is v_{2-}, v_{3+}, v_{5+}, and v_{6-}.

Possible Linked Cycles in $G[4,6,7,8,9,10]$	1-Homologous Cycle that shares an edge with a linked cycle
$(4,6,7)$, $(8,9,10)$	$(4,6,7)$
$(4,6,8)$, $(7,9,10)$	$(4,6,8)$
$(4,6,9)$, $(7,8,10)$	$(4,6,9)$
$(4,6,10)$, $(7,8,9)$	$(4,6,10)$
$(4,7,8)$, $(6,9,10)$	$(5,6,9)$
$(4,7,9)$, $(6,8,10)$	$(5,6,8)$
$(4,7,10)$, $(6,8,9)$	$(5,6,8)$
$(4,8,9)$, $(6,7,10)$	$(5,6,7)$
$(4,8,10)$, $(6,7,9)$	$(5,6,7)$
$(4,9,10)$, $(6,7,8)$	$(5,6,7)$

Table 6.

If the embedding is v_{2-}, v_{3-}, v_{5+}, and v_{6+}, since $G[7,8,9,10]$ is 0-homologous, by Lemma 2, $G[4,6,7,8,9,10]$ has a pair of links. Since $(1,2,3)$ is 1-homologous, and $(1,2,3)$ is disjoint from all the 1-homologous cycles in the second column of Table 7, Lemma 4 applies and G has a pair of links.

Possible Linked Cycles in $G[4,6,7,8,9,10]$	1-Homologous Cycle that shares an edge with a linked cycle
$(4,6,7)$, $(8,9,10)$	$(4,6,7)$
$(4,6,8)$, $(7,9,10)$	$(4,6,8)$
$(4,6,9)$, $(7,8,10)$	$(4,6,9)$
$(4,6,10)$, $(7,8,9)$	$(4,6,10)$
$(4,7,8)$, $(6,9,10)$	$(4,5,7)$
$(4,7,9)$, $(6,8,10)$	$(4,5,7)$
$(4,7,10)$, $(6,8,9)$	$(4,5,7)$
$(4,8,9)$, $(6,7,10)$	$(4,5,8)$
$(4,8,10)$, $(6,7,9)$	$(4,5,8)$
$(4,9,10)$, $(6,7,8)$	$(4,5,9)$

Table 7.

This list exhausts the possible embeddings with v_{1+} and v_{4-}. The same argument holds for the embedding has v_{1-} and v_{4+}. Thus, in every embedding of G in $\mathbb{R}P^3$, G has a triple-link.

□

Flapan, Naimi, and Pommersheim [9] showed that K_9 can be embedded 3-linklessly in $\mathbb{R}^3$, and so K_9 can be embedded 3-linklessly in $\mathbb{R}P^3$. Thus, 10 is the smallest n for which K_n is intrinsically triple-linked in $\mathbb{R}P^3$.

3 Other intrinsically triple-linked graphs in $\mathbb{R}P^3$

Proposition 9. *A graph composed of n disjoint copies of an intrinsically n-linked graph in $\mathbb{R}^3$ is intrinsically n-linked in $\mathbb{R}P^3$. In particular, three disjoint copies of intrinsically triple-linked graphs in $\mathbb{R}^3$ are intrinsically triple-linked in $\mathbb{R}P^3$*

Proof. If any of the three copies of the graph has all 0-homologous cycles, then it is crossing-change equivalent to a spatial embedding, and thus triple-linked, as its disjoint cycle pairs would have the same linking numbers as a spatial embedding. Else, all three copies have at least one 1-homologous cycle. Then we have three disjoint 1-homologous cycles, and thus have a triple-link. □

As shown above, K_{10} is an example of a one-component graph that is intrinsically triple-linked in $\mathbb{R}^3$. In the following section, we will exhibit two examples of minor-minimal intrinsically triple-linked graphs, each comprised of two components, that

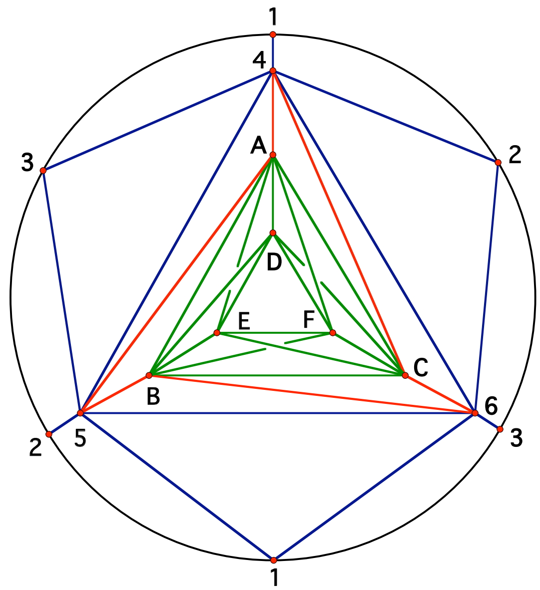

Figure 3: A 3-linkless embedding of K_6 connected to K_6 along a 6-cycle in $\mathbb{R}P^3$.

are intrinsically triple-linked in $\mathbb{R}^3$. The question remains whether there exists a minor-minimal intrinsically triple-linked graph of three components in $\mathbb{R}P^3$.

We will use the following theorem:

Theorem 10. [2] *Let G be a graph containing two disjoint graphs from the Petersen family, G_1 and G_2 as subgraphs. If there are edges between the two subgraphs G_1 and G_2 such that the edges form a 6-cycle with vertices that alternate between G_1 and G_2, then G is minor-minimal intrinsically triple-linked in $\mathbb{R}^3$.*

If G_1 and G_2 are isomorphic to K_6, this result does not hold in $\mathbb{R}P^3$, as seen in Figure 3.

Proposition 11. *If G_1 and G_2 are disjoint copies of K_6 connected to K_6 along a 6-cycle with vertices that alternate between the copies of K_6, then $G = G_1 \sqcup G_2$ is minor-minimal intrinsically triple-linked in $\mathbb{R}P^3$.*

Proof. Embed G in $\mathbb{R}P^3$. If G_1 or G_2 has all 0-homologous cycles, G will have a triple-link since K_6 connected to K_6 along a 6-cycle with vertices that alternate between the copies of K_6 is triple-linked in $\mathbb{R}^3$. Thus, G_1 and G_2 each have a 1-homologous cycle. Let G_1 be a graph on the vertex set $\{1, 2, 3, 4, 5, 6, A, B, C, D, E, F\}$ where $G[1, 2, 3, 4, 5, 6]$ and $G[A, B, C, D, E, F]$ are the copies of K_6 and the connecting edges are $(4, A)$, $(4, C)$, $(5, A)$, $(5, B)$, $(6, B)$, and $(6, C)$. Up to isomorphsim, there are five

3-cycle equivalence classes in G_1. Consider $S = \{(1,2,3),(1,2,4),(1,4,5),(4,5,6),(4,5,A)\}$, which contains one representative from each 3-cycle class. We assume, without loss of generality, one cycle in S is 1-homologous.

Consider $G[A,B,C,D,E,F]$. If there is a one homologous cycle in $G[B,C,E,F]$ then this cycle will link with the cycle in S that is 1-homologous. Since the cycle from S links with the 1-homologous cycle in G_2, we have a triple-link in G. Thus, we assume every cycle in $G[B,C,E,F]$ is 0-homologous and so $G[A,B,C,D,E,F]$ has a pair of linked cycles. By the pigeon-hole principle, at least two edges connecting vertices from the set $\{A,B,C\}$ are in a linked cycle in $G[A,B,C,D,E,F]$, so, without loss of generality, we may assume v_A and v_B are in one cycle. If the 1-homologous cycle is in the subset $S_1 = \{(1,2,3),(1,2,4),(1,4,5),(4,5,6)\}$, then there are disjoint edges from the 6-cycle that connect the cycle from S_1 to the cycle containing v_A and v_B. So, by Lemma 4, G has a triple-link.

If $(4,5,A)$ is the 1-homologous cycle, consider $G[1,2,3,4,5,6]$. If there is a one homologous cycle in $G[1,2,3,6]$ then this cycle will link with $(4,5,A)$ and the 1-homologous cycle in G_2, so G will have a triple-link. Else, $G[1,2,3,4,5,6]$ has a pair of linked cycles. By the pigeon-hole principle, at least two vertices in the set $\{4,5,6\}$ are in a linked cycle within the embedding of one copy of K_6. Similarly, at least two vertices of $\{A,B,C\}$ are in a linked cycle in the other copy of K_6. As a result of the 6-cycle, there are two disjoint edges between the cycles and Lemma 4 then applies and G is triple-linked.

To see G is minor-minimal with respect to intrinsic triple-linking in $\mathbb{R}P^3$, embed G so that G_1 is embedded as in the drawing in Figure 3 and G_2 is contained in a sphere that lies in the complement of G_1. Therefore, G_1 does not have any triple-links and no cycle in G_1 is linked with a cycle in G_2. Without loss of generality, if we delete an edge, contract an edge or delete any vertex on G_2, it will have an affine linkless embedding. Thus, we can re-embed G_2 within the sphere in each case. Thus, G is minor-minimal for intrinsic triple-linking.

$\square$

Theorem 12. [2] *Let G be a graph formed by identifying an edge of K_7 with an edge from another copy of K_7. Then G is intrinsically triple-linked in $\mathbb{R}^3$.*

If G is isomorphic to K_7 connected to K_7 along an edge, this result does not hold in $\mathbb{R}P^3$, as seen in Figure 4.

We will need the following lemma:

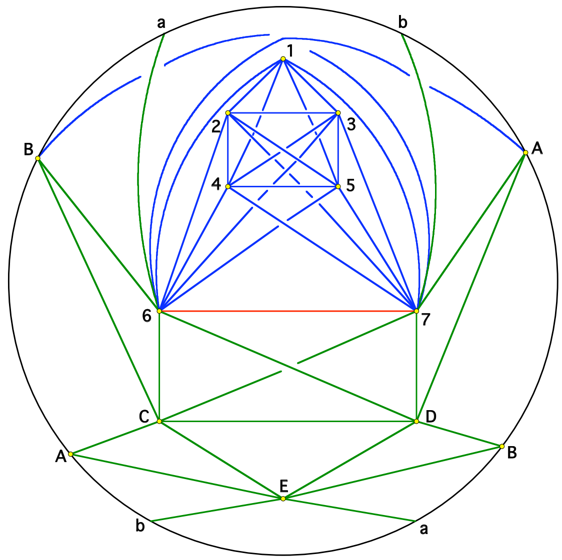

Figure 4: A 3-linkless embedding of K_7 connected to K_7 along an edge in $\mathbb{R}P^3$.

Lemma 13. [3] *Let P be a Petersen-family graph and v be a vertex of P. If every cycle of $P \setminus \{v\}$ is 0-homologous in an embedding $f : P \to \mathbb{R}P^3$, then $f(P)$ contains a non-trivial link.*

Proposition 14. *If G_1 and G_2 are disjoint copies of K_7 connected to K_7 along an edge, then $G = G_1 \sqcup G_2$ is intrinsically triple-linked in $\mathbb{R}P^3$.*

Proof. If G_1 or G_2 have all 0-homologous cycles, G will have a triple-link since K_7 connected to K_7 along an edge is triple-linked in $\mathbb{R}^3$. Thus, G_1 and G_2 each have a 1-homologous cycle. Let G_1 be a graph on the vertex set $\{1, 2, 3, 4, 5, 6, 7, A, B, C, D, E\}$ where $G[1, 2, 3, 4, 5, 6, 7]$ and $G[6, 7, A, B, C, D, E]$ are the copies of K_7 and the connecting edge is $(6, 7)$. Up to isomorphsim, there are three 3-cycle equivalence classes in G_1. We consider $S = \{(1, 2, 3), (1, 2, 7), (1, 6, 7)\}$, which contains one representative from each 3-cycle class. We can assume, without loss of generality, at least one cycle of S is 1-homologous.

Case 1: Let $(1, 2, 3)$ be a 1-homologous cycle in G_1. Then $(1, 2, 3)$ links with the 1-homologous cycle in G_2. Consider $G[A, B, C, D, E, 6]$. If $G[A, B, C, D, E, 6]$ has a 1-homologous cycle, then there are three disjoint 1-homologous cycles, so we assume $G[A, B, C, D, E, 6]$ must be 0-homologous and so $G[A, B, C, D, E, 6]$ has a pair of linked cycles. Lemma 4 applies with v_7 connecting to the cycle that uses v_6, following the proof in [2].

Case 2: Let $(1, 2, 7)$ be a 1-homologous cycle in G_1. $(1, 2, 7)$ links with the 1-homologous cycle in G_2. Consider $G[A, B, C, D, E, 6]$. If $G[A, B, C, D, E, 6]$ has a

1-homologous cycle, then there are three disjoint 1-homologous cycles, so we assume $G[A,B,C,D,E,6]$ must be 0-homologous and so $G[A,B,C,D,E,6]$ has a pair of linked cycles. Lemma 4 applies with v_7 connecting to the cycle that uses v_6, following the proof in [2].

Case 3: Let $(1,7,6)$ be a 1-homologous cycle in G_1. $(1,7,6)$ links with the 1-homologous cycle in G_2. Consider $G[A,B,C,D,E,6]$. If $G[A,B,C,D,E]$ has a 1-homologous cycle, then there are three disjoint 1-homologous cycles, so we assume $G[A,B,C,D,E]$ must be 0-homologous. Then, by Lemma 13, $G[A,B,C,D,E,6]$ has a pair of linked cycles. Lemma 4 applies with v_7 connecting to the cycle with v_6, following the proof in [2].

□

We note that if K_7 connected to K_7 along an edge is minor-minimal with respect to triple-linking in $\mathbb{R}^3$, then we would also have that two disjoint copies of K_7 connected to K_7 along an edge is minor-minimal intrinsically triple-linked in $\mathbb{R}P^3$. However, the minor-minimality of this graph is still unknown in $\mathbb{R}^3$.

We also note that $G(n)$, as defined in [7], is a one-component minor-minimal intrinsically $(n+1)$-linked graph in $\mathbb{R}P^3$, by the same arguement given in [7], since $K_{4,4}-e$ is intrinsically linked in both $\mathbb{R}^3$ and $\mathbb{R}P^3$.

4 Graphs with linking number ≥ 1 in $\mathbb{R}P^3$

In $\mathbb{R}P^3$, there are intrinsically linked graphs for which there exists an embedding in which every pair of disjoint cycles has linking number less than 1. Work has been done in $\mathbb{R}^3$ to find graphs containing disjoint cycles with large linking number in every spatial embedding. Using the fact that K_{10} is triple-linked in $\mathbb{R}^3$, Flapan [6] showed that every spatial embedding of K_{10} contains a two-component link $L \cup J$ such that, for some orientation, $lk(L,J) \geq 2$. A similar argument using Theorem 8 yields the following proposition.

Proposition 15. *Every projective embedding of K_{10} contains a two-component link $L \cup J$ such that, for some orientation, $lk(L,J) \geq 1$.*

It remains an open question to determine if 10 is that smallest number for which this property holds. At this point, we know the smallest n is such that $7 < n \leq 10$.

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